# Functional Relation of interquark potential with interquark distance

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## Abstract

The functional relation between interquark potential and interquark distance is explicitly derived by considering Nambu-Goto action in the $AdS \times S^5$ background. It is also shown that similar relation holds in the general background. The implication of this relation in confinement is briefly discussed.

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$AdS/CFT$ correspondence [1] enables us to understand the quantum phenomena in large $N$ limit of $\mathcal{N} = 4$ super Yang-Mills theory from the classical string description [2,3] in the $AdS \times S^5$ background. Especially, the finite temperature effect in the side of gauge theories is discussed in detail [4] and wilson loop is calculated at zero temperature [5,6] and finite temperature [7,8]. The main differences of finite temperature case, in our opinion, from the zero temperature case are (1) the presence of a maximal separation distance which makes the dependence of the interquark distance of the minimum point of string to be multi-valued function; (2) the appearance of cusp (or bifurcation point) in the graph of interquark potential-vs-interquark distance [7].

These facts strongly suggests that there is a hidden functional relation between these quantities as we learned from the statistical mechanics.

In this letter we will derive this relation explicitly in the $AdS$ background. Also, it will be shown that similar relation[1] holds at arbitrary background.

We start with Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{det G_{MN} \partial_\alpha X^M \partial_\beta M^N} \tag{1}$$

in the Euclidean Schwarzschild-$AdS$ background

$$ds_E^2 = \alpha' \left[ \frac{U^2}{R^2} \left( f(U) dt^2 + dx_i dx_i \right) + \frac{R^2 f(U)^{-1}}{U^2} dU^2 + R^2 d\Omega_5^2 \right]. \tag{2}$$

Here, $x_i (i = 1, 2, 3)$ is $D3$-brane coordinate and $R$ is radius of $AdS$ and $S^5$. $f(U)$ is defined by $f(U) = 1 - U_T^4/U^4$ and temperature is given by $T = U_T/(\pi R^2)$. After identifying $\tau = t$ and $\sigma = x$, it is easy to show that for the static case $S_{NG}$ becomes

$$S_{NG} = \frac{\tilde{\tau}}{2\pi} \int dx \sqrt{U'^2 + \frac{U^4 - U_T^4}{R^4}} \tag{3}$$

where prime means the differentiation with respect to $x$ and $\tilde{\tau}$ is whole Euclidean time interval. Using the constant of motion

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[1] At arbitrary background this is a relation between string energy and the distance of string ends at the boundary.

$$\frac{U^4 - U_T^4}{\sqrt{U'^2 + \frac{U^4 - U_T^4}{R^4}}} = const \equiv R^2 \sqrt{U_0^4 - U_T^4} \tag{4}$$

where $U_0$ is minimum point of string configuration, one can explicitly derive the static solution of $S_{NG}$ in terms of elliptic functions;

$$\begin{aligned} x = \frac{R^2\sqrt{U_0^2 - U_T^2}}{2\sqrt{2}U_0 U_T} \Bigg[ & F\left(\sin^{-1}\frac{\sqrt{(U^2 - U_0^2)(U^2 - U_T^2)}}{U^2 - U_0 U_T}, \frac{U_0 + U_T}{\sqrt{2(U_0^2 + U_T^2)}}\right) \\ & - F\left(\sin^{-1}\frac{\sqrt{(U^2 - U_0^2)(U^2 - U_T^2)}}{U^2 + U_0 U_T}, \frac{U_0 - U_T}{\sqrt{2(U_0^2 + U_T^2)}}\right)\Bigg] \end{aligned} \tag{5}$$

where $F$ is elliptic function of the first kind. Since interquark distance $L$ is defined by the distance of different ends of string at the $AdS$ boundary($U \to \infty$), it is easy to compute $L$ from (5);

$$L = \frac{R^2}{\sqrt{2}U_T}\frac{\sqrt{a^2 - 1}}{a}\left[K(f_1(a)) - K(f_2(a))\right] \tag{6}$$

where $a = U_0/U_T$, $K$ is complete elliptic function, and $f_1(a)$ and $f_2(a)$ are

$$\begin{aligned} f_1(a) &= \frac{a+1}{\sqrt{2(a^2+1)}} \\ f_2(a) &= \frac{a-1}{\sqrt{2(a^2+1)}}. \end{aligned} \tag{7}$$

It is important to note that $f_1(a)$ and $f_2(a)$ are complementary modulus to each other in the sense of $f_1(a)^2 + f_2(a)^2 = 1$.

Energy of solution (5) which is interpreted as a interquark potential in the context of $AdS/CFT$ correspondence is straightforwardly derived using the constant of motion (4);

$$E_{q\bar{q}} = \frac{U_T}{\pi}\lim_{\Lambda\to\infty}\int_a^\Lambda dy \sqrt{\frac{y^4 - 1}{y^4 - a^4}} \tag{8}$$

where $\Lambda$ is cutoff parameter. To regularize $E_{q\bar{q}}$ we have to substract the quark mass [1] and finite form of interquark potential is

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi}\lim_{\Lambda\to\infty}\left[\int_a^\Lambda dy \sqrt{\frac{y^4 - 1}{y^4 - a^4}} - (\Lambda - 1)\right]. \tag{9}$$

This is also computed in terms of elliptic functions;

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi}\left[1 + \sqrt{\frac{a^2+1}{2}}\left[\frac{a-1}{2a}K(f_1(a)) + \frac{a+1}{2a}K(f_2(a)) - E(f_1(a)) - E(f_2(a))\right]\right] \quad (10)$$

where $E$ is complete elliptic function.

The $L$-dependence of $E_{q\bar{q}}^{(Reg)}$ at various temperatures is ploted at Fig. 1. As we mentioned, the appearance of cusps at finite temperature strongly suggests that there exists a functional relation between $E_{q\bar{q}}^{(Reg)}$ and $L$ from our experience of van der Waals gas. To find this relation it is more convenient to introduce a modulus $\kappa = f_2(a) = (a-1)/\sqrt{2(a^2+1)}$ explicitly. Since $a \geq 1$[2], $\kappa$ is defined in the region $0 \leq \kappa \leq 1/\sqrt{2}$. Inverting it yields

$$a = \frac{1+2\kappa\kappa'}{1-2\kappa^2} \quad (11)$$

where $\kappa' \equiv \sqrt{1-\kappa^2}$. Then $L$ and $E_{q\bar{q}}^{(Reg)}$ are expressed in terms of $\kappa$ and $\kappa'$ as following;

$$L = \frac{R^2}{\sqrt{2}U_T}\frac{2\sqrt{\kappa\kappa' + 2\kappa^2\kappa'^2}}{1+2\kappa\kappa'}\left[K(\kappa') - K(\kappa)\right] \quad (12)$$

$$E_{q\bar{q}}^{(Reg)} = \frac{U_T}{\pi}\left[1 + \frac{\sqrt{1+2\kappa\kappa'}}{1-2\kappa^2}\left[\frac{\kappa(\kappa+\kappa')}{1+2\kappa\kappa'}K(\kappa') + \frac{1+\kappa(\kappa'-\kappa)}{1+2\kappa\kappa'}K(\kappa) - E(\kappa') - E(\kappa)\right]\right].$$

Since statistical relation is usually realized at the first derivative level, we compute $dL/d\kappa$ and $dE_{q\bar{q}}^{(Reg)}/d\kappa$ whose explicit forms are

$$\frac{dL}{d\kappa} = \frac{R^2}{\sqrt{2}U_T}\frac{1}{\kappa'(1+2\kappa\kappa')\sqrt{\kappa\kappa' + 2\kappa^2\kappa'^2}} \times\left[(1+4\kappa^3\kappa')K(\kappa') + (1+4\kappa\kappa'^3)K(\kappa) - 2(1+2\kappa\kappa')\left(E(\kappa)+E(\kappa')\right)\right] \quad (13)$$

$$\frac{dE_{q\bar{q}}^{(Reg)}}{d\kappa} = \frac{U_T}{\pi}\frac{1}{\kappa'(\kappa'^2-\kappa^2)^2\sqrt{1+2\kappa\kappa'}} \times\left[(1+4\kappa^3\kappa')K(\kappa') + (1+4\kappa\kappa'^3)K(\kappa) - 2(1+2\kappa\kappa')\left(E(\kappa)+E(\kappa')\right)\right].$$

It is worthwhile noting that the coefficients of complete elliptic functions in the brackets of $dL/d\kappa$ and $dE_{q\bar{q}}^{(Reg)}/d\kappa$ coincide with each other. This means $dE_{q\bar{q}}^{(Reg)}/dL$ is independent of the complete elliptic function;

[2]This is a consequence of Ref. [9] that D-branes are located at the horizon. Although we have somewhat different opinion, we will follow this condition in this letter. Our opinion will be discussed elsewhere.

$$\frac{dE_{q\bar{q}}^{(Reg)}}{dL} = \frac{\sqrt{U_0^4 - U_T^4}}{2\pi R^2}. \tag{14}$$

The right-hand side of Eq.(14) is the regularized energy of constant solution $U = U_0$; $E^{(Reg)}(U_0)$. So, we obtain finally the functional relation

$$\frac{dE_{q\bar{q}}^{(Reg)}}{dL} = E^{(Reg)}(U_0). \tag{15}$$

Actually, this is very similar to the relation of classical Euclidean point particle: $dS_E/dP = \mathcal{E}$ where $S_E$, $P$, and $\mathcal{E}$ are Euclidean action, period and energy of particle.

It is interesting to compare Eq. (15) with a case of van der Waals gas. Our plot of $E_{q\bar{q}}^{(Reg)}$-vs-$L$ is completely analogous to the plot of enthalpy-vs-pressure of the gas whose equation of state plotted as pressure-vs-volume corresponds to the plot of $L$-vs-$U_0$. Hence, the plot of $E_{q\bar{q}}^{(Reg)}$-vs-$L$ is completely determined from the plot of $L$-vs-$U_0$ up to the constant. If the figure of $L$-vs-$U_0$ does have $n$ extremum points, the figure of $E_{q\bar{q}}^{(Reg)}$-vs-$L$ has $n$ bifurcation points.

To get more generality we consider the Nambu-Goto action (1) in the arbitrary space-time background. Then the static action will be reduced generally to

$$S_{NG} = \frac{\tilde{\tau}}{2\pi} \int dx \sqrt{G_1(U)U'^2 + G_2(U)} \tag{16}$$

where $G_1(U)$ and $G_2(U)$ are metric-dependent functions. The constant of motion in this case is

$$\frac{G_2(U)}{\sqrt{G_1(U)U'^2 + G_2(U)}} = const \equiv \sqrt{G_2(U_0)}. \tag{17}$$

Hence, we choosed $U = U_0$ as an extreme point in the string configuration. We assume $G_1(U_0) \neq 0$ and $G_2(U_0) \neq 0$. Although this assumption excludes particular space-time geometry, we don't think these assumption restricts our general argument crucially. From the constant of motion it is easy to show that the solution of the static Nambu-Goto action (16) obeys the integral equation;

$$x = \int_{U_0}^{U} dU \sqrt{\frac{G_1(U)G_2(U_0)}{G_2(U)[G_2(U) - G_2(U_0)]}} \tag{18}$$

From this integral equation it is straightforward to show that the distance $L$ between string ends at $U \to \infty$ and string energy $\mathcal{E}$ become

$$L = 2\sqrt{G_2(U_0)} \int_{U_0}^{\infty} dU \sqrt{\frac{G_1(U)}{G_2(U)[G_2(U) - G_2(U_0)]}} \tag{19}$$
$$\mathcal{E} = \frac{1}{\pi} \int_{U_0}^{\infty} dU \sqrt{\frac{G_1(U)G_2(U)}{G_2(U) - G_2(U_0)}}.$$

Using Leibnitz rule one can prove directly

$$\frac{d\mathcal{E}}{dU_0} \approx -\frac{\sqrt{G_1(U_0)G_2(U_0)}}{\pi} \lim_{U \to U_0} \frac{1}{\sqrt{G_2(U) - G_2(U_0)}} \tag{20}$$
$$\frac{dL}{dU_0} \approx -2\sqrt{G_1(U_0)} \lim_{U \to U_0} \frac{1}{\sqrt{G_2(U) - G_2(U_0)}}$$

which recovers our relation;

$$\frac{d\mathcal{E}}{dL} = \frac{1}{2\pi}\sqrt{G_2(U_0)} \equiv \mathcal{E}^{(Reg)}(U_0). \tag{21}$$

In our opinion, this kind of relation is not restricted only to Nambu-Goto action. We think there exists a similar relation in Born-Infeld action, which is under study.

Now, we discuss the confinement briefly using our relation (15). Since the confinement and deconfinement depends on the $L$-dependence of $E_{q\bar{q}}^{(Reg)}$, say $E_{q\bar{q}}^{(Reg)} \propto L^{\alpha}$ where $\alpha \geq 1$ and $\alpha \leq -1$ represent confinement phase and deconfinement phase respectively, it is important to draw a figure of $dE_{q\bar{q}}^{(Reg)}/dL$ with respect to $L$ which is given at Fig. 2. Each line in Fis. 2 consists of confining and deconfining parts. Fig. 2 also indicates that as temperature increases, the confining force becomes strong(large $\alpha$) while available $L$ is reduced, which is physically predictable. At low temperature $\alpha$ approaches to 1 and hence the area law of wilson loop is recovered [10]. It should be noted that the potential energy of confining part is always larger than that of the deconfining part. This means our explanation on confinement is still incomplete. Full understanding of the quark confinement seems to require more deep understanding of relation between the gauge theories and statical physics like black hole.

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## FIGURES

FIG. 1. $L$-dependence of $E_{q\bar{q}}^{(Reg)}$. The appearance of the cusps in this figure strongly suggest that there exists an hidden relation between $E_{q\bar{q}}^{(Reg)}$ and $L$.

FIG. 2. $L$ dependence of $dE_{q\bar{q}}^{(Reg)}/dL$. Each line consists of confining(lower) and deconfining(upper) parts. This figure indicates that more strong confining force is required at high temperature. At low temperature the area law of wilson loop is recovered.

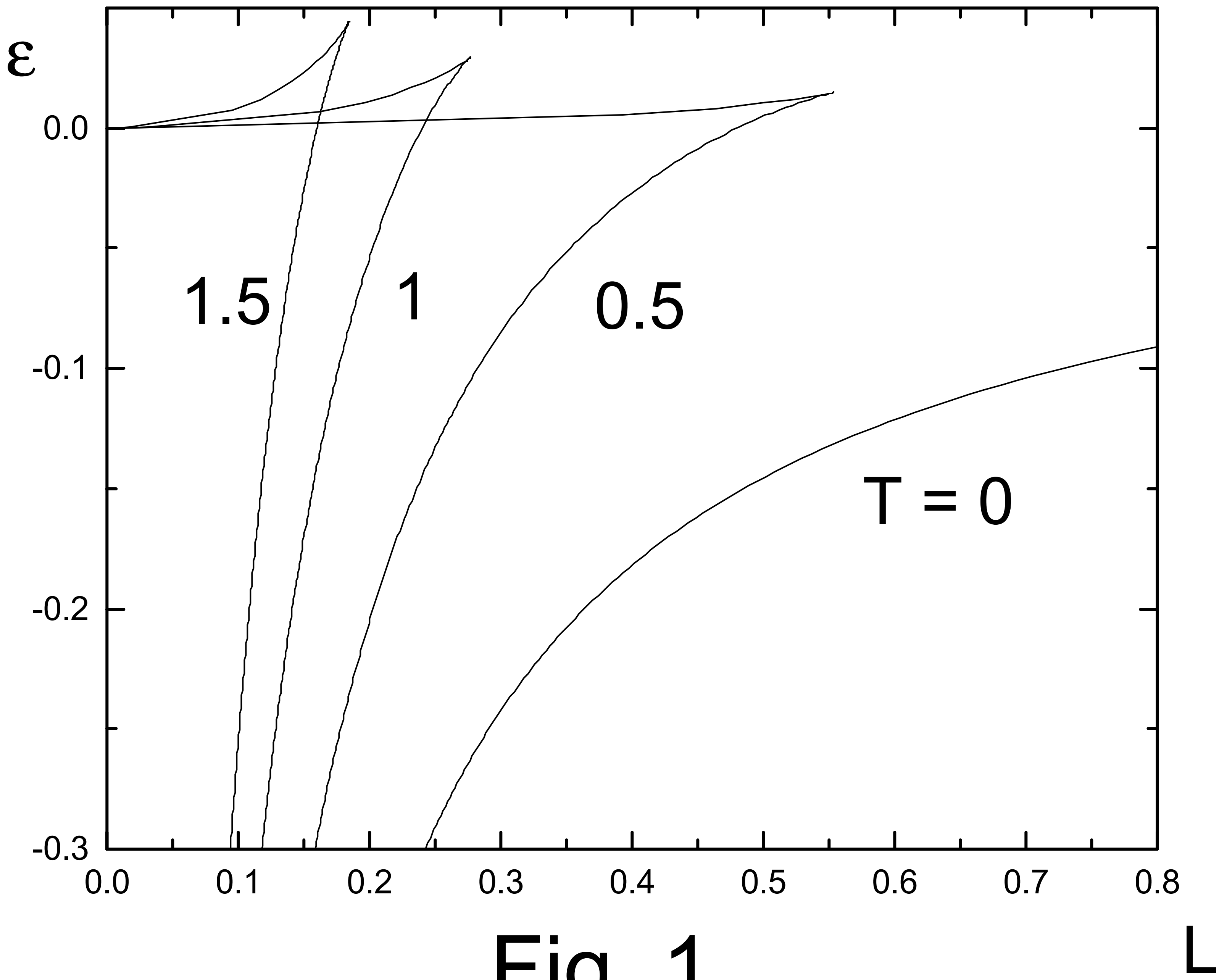


Fig. 1

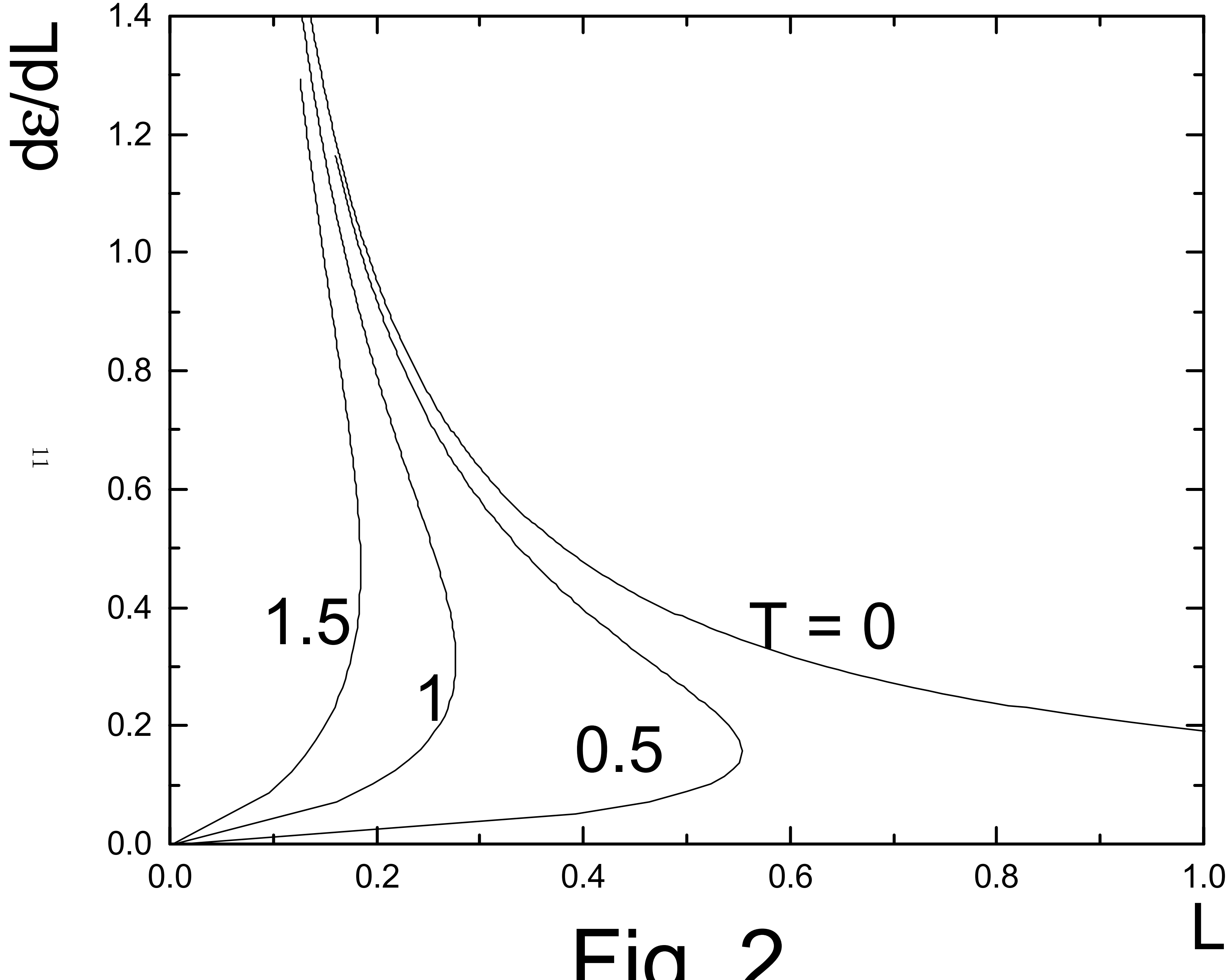


Fig. 2